[P3] It has been claimed that the energy of an Ising spin system approaches the ground state energy as the temperature goes to zero:

$$\lim_{T\to 0} E(T) = \min_{\sigma} H(\sigma).$$

We have assumed that for non-zero temperature, the system is described by the usual Boltzmann ensemble. Prove this claim. (2 P.)

[P4] Computer scientists like to think in terms of *decision problems*, e.g.

IsingGroundState$_{\mathrm{D}}$: Given coupling matrix J and number k,
is there a configuration with energy smaller than k?

For us, it is more natural to consider the *quantitative* problem

IsingGroundState$_{\mathrm{Q}}$: Given coupling matrix J, what is the ground state energy?

Prove that the two formulations are computationally equivalent in the sense that there is a polynomial-time algorithm for IsingGroundState$_{\mathrm{D}}$ if and only if there is one for IsingGroundState$_{\mathrm{Q}}$. (3 P.)

[P5] One way of showing that finding the ground state of the Ising model is NP-hard is by way of the following sequence of reductions:

(any $L \in$ NP) $\leq_p$ Sat $\leq_p$ 3Sat $\leq_p$ Min-2-Sat $\leq_p$ MaxCut $\leq_p$ IsingGroundState.

We will consider the various reductions one by one. Here, we focus on 3Sat $\leq_p$ Min-2-Sat.

Recall the underlying definitions. A *Boolean formula* is a mathematical formula consisting of variables u_i taking values in $\{0, 1\}$ ("true", "false"), and the symbols $\neg, \wedge, \vee$ ("not", "and", "or") to be interpreted in the usual way. (Example: $(u_1 \wedge u_2) \vee (\neg u_1 \wedge \neg u_2)$ evaluates to 1 if and only if $u_1 = u_2$.) A formula made up only of one variable or its negation is a *literal*. A *clause* is a sequence of literals separated by $\vee$'s. (Example: $u_1, u_1 \vee u_2, u_1 \vee (\neg u_1) \vee u_4$ are clauses). A k-clause is a clause containing k literals. An *assignment* is a choice of values $\{0, 1\}$ for each variable u_i. An assignment *satisfies* a Boolean formula, if that formula evaluates to 1 for the given assignment. We'll prove in the lecture that the following problem is NP hard:

3Sat: Given a list of 3-clauses,
decide whether there is an assignment which simultaneously satisfies all clauses.

Here, the task is to prove that the problem

Max-2-Sat: Given a list of 2-clauses and integer k,
decide whether there is an assignment which satisfies k of them.

is NP hard, by reduction from 3SAT.

(1) Let a, b, c, d be literals. Show that the 3-clause $(a \vee b \vee c)$ is satisfiable if 7 of the following 2-clauses are satisfiable:

$$(a \vee a), (b \vee b), (c \vee c), (d \vee d), (\neg a \vee \neg b), (\neg a \vee \neg c), (\neg b \vee \neg c), (a \vee \neg d), (b \vee \neg d), (c \vee \neg d).$$

Show that no more than 7 of these clauses can ever be simultaneously satisfied. Further, show that if $(a \vee b \vee c)$ is not satisfiable, then it is not possible to satisfy more than 6 of the clauses above.

(2) Based on the previous result, show that a list of m 3-clauses can be mapped to a list of $10m$ 2-clauses, such that the there is a satisfying assignment for all the 3-clauses if and only if at least $7m$ of the 2-clauses can be satisfied simultaneously. (5 P.)